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Boundary integral equation method for general viscoelastic analysis

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Abstract

From basic assumptions of viscoelastic constitutive relations and weight residual techniques a Boundary Element procedure is achieved for both Kelvin and Boltzmann models. Imposing spatial approximations and adopting convenient kinematical relations for strain velocities, a system of time differential equations is achieved. This system is solved adopting linear approximations for displacements, resulting in a time marching methodology. This approach avoids the use of relaxation functions and makes easier changes in boundary conditions along time, natural or essential. An important feature of the resulting technique is the absence of domain discretizations, which simplify the treatment of problems involving infinite domains (tunnels and cavities inside the soil). Some examples are shown in order to demonstrate the accuracy and stability of the technique when compared to analytical solutions. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

In some recent works (see e.g. Mesquita et al. (2001) and Mesquita and Coda (2002, 2001)) the authors developed a new time marching process for both Finite Element Method (FEM) and Boundary Element Method (BEM) to solve viscoelastic problems. These formulations are based on the differential constitutive relation for Kelvin and Boltzmann viscoelastic models. They produce time differential systems of equations, solved by an appropriate time marching process. The resulting algorithms are able to solve static viscoelastic problems with any load time dependence and boundary conditions. However, at that time, the BEM formulation was not completely developed, i.e., it was necessary to perform domain integrals (using internal cells) in order to consider the viscous effects.

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The novelty and main objective of this paper is avoiding internal cells, resulting in a viscoelastic Boundary Element formulation performing discretizations only at the boundary of the analysed body. This improvement is very important because it reduces the amount of variables to be computed. It makes easy the treatment of infinite and semi-infinite viscoelastic bodies under loading or material extraction.

Another feature of the developed formulation is that, for Boltzmann model, the total time-dependent displacements and stresses are achieved directly from the time marching process, not by summation of the instantaneous and viscous uncoupled parts (see e.g. Munaiar Neto (1998) and Fairbaim et al. (1995)).

The formulation proposed here is quite different from the ones usually found in the literature. The most of the works developed so far as follows, basically, three main procedures. All of them are based on relaxation functions, providing a time-dependent constitutive relation (see e.g. Lemaitre and Chaboche (1990), Flügge (1967), Sobotka (1984) and Christensen (1982)).

The first procedure is based on the use of relaxation functions together with a convenient incremental scheme, where the convolutional aspect of the viscous behaviour is transformed into discrete contributions properly added to the elastic response (see e.g. Carpenter (1972), Chen et al. (1993), Chen and Lin (1982), Argyris et al. (1979), Sensale et al. (2001), Liu et al. (2000) and Liu (1994)). The second available formulation follows the same scheme applied to viscoplastic analysis (see e.g. Munaiar Neto (1998), Fairbaim et al. (1995), Perzyna (1963), Owen and Damjanic (1982) and Argyris et al. (1991)), in which the viscous characteristics are incorporated to the effective stress–strain relation by means of relaxation functions, leading to incremental techniques.

The third one provides a Laplace–Carson transformation of the viscoelastic problem to an equivalent elastic one. After solving the transformed problem, a numerical inversion is performed recovering the desired time domain behaviour (see e.g. Lemaitre and Chaboche (1990)).

Regarding the mentioned procedures, some brief remarks can be made. The last technique is appropriate to solve problems in which the nature of boundary conditions does not change along time. The first and second procedures are based on quasi-static incremental schemes where the time behaviour of the solution is recovered by stress decay, therefore, imposing external loads with arbitrary time dependence presents some difficulties.

The main difference of the proposed scheme and the ones available in literature is the time solution. The use of incremental methods based on relaxation functions assumes a known behaviour (usually constant) of the total stress during a load increment. From this assumption, one solves locally the time differential stress/strain relation achieving the viscous contribution to the body behaviour. This contribution is applied on the equilibrium equation as a correction term. The proposed formulation assumes a kinematical relation for strain velocity, i.e., relates strain velocity to material velocity. From this relation a global time differential system of equation is achieved and properly solved.

At the end of this paper, an example section is provided. Various examples are shown in order to demonstrate the accuracy and stability of the formulation. Analytical solutions are taken for comparison because they are the natural accuracy reference parameter. It is not possible, from the consulted literature, to compare the performance of this technique with others, because no data about time processing or stability are given in references (see e.g. Sensale et al. (2001), Liu et al. (2000) and Liu (1994)).

Along all text Einstein notation is adopted.

2. Basic relations for viscoelasticity

This section is divided into two main parts, one related to the Kelvin model and the other related to the Boltzmann standard relations.

2.1. Kelvin model

Using rheological models defined in the uniaxial space is the usual way adopted to describe the visco-elastic behaviour of solids. A simple representation, very often adopted to describe this kind of behaviour, is the Kelvin–Voigt viscoelastic, Fig. 1. The understanding of this simple model is a basic step to the development of more complicated ones, as for example the Boltzmann model described in the next item.

From Fig. 1, the following relations are stated:

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} = \varepsilon_{ij}^{v}, \tag{1}$$

$$\sigma_{ij} = \sigma_{ij}^{e} + \sigma_{ij}^{v}, \tag{2}$$

where ε and σ are the strain and stress tensors; the Cartesian co-ordinates are represented by subscripts i and j, while the superscript v and e represent viscous and elastic parts, respectively.

The elastic stress can be written in terms of strain components, as follows:

$$\sigma_{ij}^{e} = C_{ij}^{lm}\varepsilon_{lm}^{e} = C_{ij}^{lm}\varepsilon_{lm}. \tag{3}$$

Similarly, the following relation gives the viscous stress components:

$$\sigma_{ij}^{v} = \eta_{ij}^{lm}\dot{\varepsilon}_{lm}^{v} = \eta_{ij}^{lm}\dot{\varepsilon}_{lm}. \tag{4}$$

In Eqs. (3) and (4), C_{ij}^{lm} and η_{ij}^{lm} contain the elastic compliance factors and the viscous constitutive parameters, respectively, defined as follows:

$$C_{ij}^{lm} = \lambda\delta_{ij}\delta_{lm} + \mu(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}), \tag{5}$$

$$\eta_{ij}^{lm} = \theta_{\lambda}\lambda\delta_{ij}\delta_{lm} + \theta_{\mu}\mu(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}), \tag{6}$$

where λ and μ are Lamè's constants, given by:

$$\lambda = \frac{vE}{(1+v)(1-2v)}, \tag{7}$$

$$\mu = G = \frac{E}{2(1+v)}, \tag{8}$$

in which E and v are Young's modulus and Poisson ratio, respectively, while θ_{λ} and θ_{μ} are the hydrostatic and deviatoric viscosity coefficients.

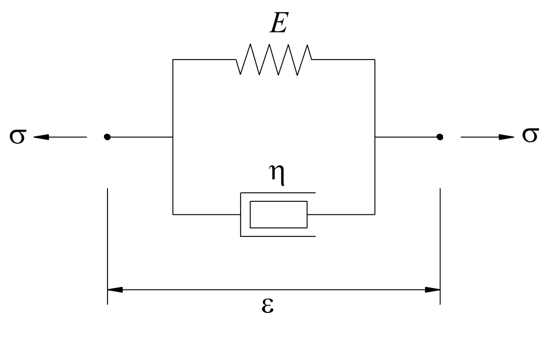

Fig. 1. Kelvin–Voigt viscoelastic model (uniaxial representation).

Replacing Eqs. (3) and (4) into Eq. (2) gives

$$\sigma_{ij} = C_{ij}^{lm}\varepsilon_{lm} + \eta_{ij}^{lm}\dot{\varepsilon}_{lm}. \tag{9a}$$

In this work, a further simplification is assumed, i.e., $\theta_\lambda = \theta_\mu = \gamma$, in order to obtain only boundary values at the integral equations, see Section 3.

$$\sigma_{ij} = C_{ij}^{lm}\varepsilon_{lm} + \gamma C_{ij}^{lm}\dot{\varepsilon}_{lm}. \tag{9b}$$

Viscous effects should be incorporated into the global equilibrium equation of the body taking into account the non-local characteristics of the stresses. Moreover, the viscous characteristics of the body must satisfy boundary conditions together with the elastic ones.

In order to fulfil these requirements one can write properly the actual equilibrium equation for an infinitesimal part of a general viscoelastic body, as follows:

$$\sigma_{ij,i} + b_j = \rho\ddot{u}_j + c\dot{u}_j \tag{10}$$

or

$$\sigma_{ij,i}^{\mathrm{e}} + \sigma_{ij,i}^{\mathrm{v}} + b_j = \rho\ddot{u}_j + c\dot{u}_j, \tag{11}$$

where b_j is the body force acting in j direction.

Note that Eq. (11) exhibits explicitly the viscous stress term which plays an important role in the body equilibrium. As in this work the dynamic effects, inertia forces and friction, will not be considered, expression (11) should be rewritten as:

$$\sigma_{ij,i}^{\mathrm{e}} + \sigma_{ij,i}^{\mathrm{v}} + b_j = 0. \tag{12}$$

2.2. Boltzmann model

Another representation employed to describe the mechanical behaviour of viscoelastic materials, stress/strain constitutive relation, is the so-called standard Boltzmann model. This model is more general than the previous one, and can be described in a uniaxial representation as illustrated in Fig. 2.

This model is represented by a serial arrangement of a Kelvin–Voigt model and an elastic relation. It can reproduce both the instantaneous and the viscous behaviour of a specific material.

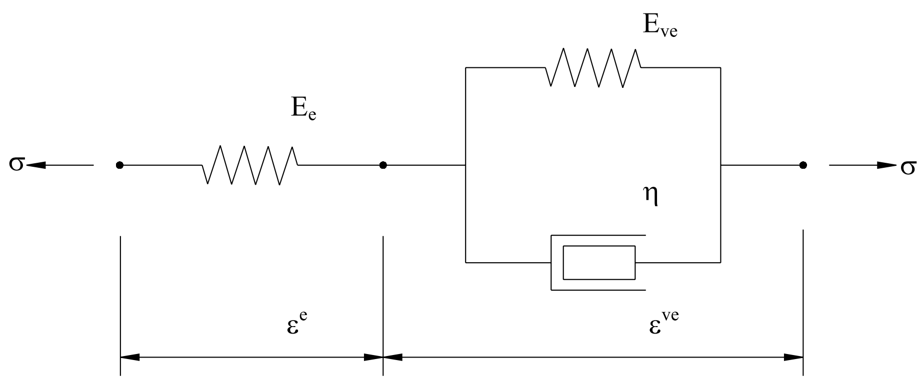

Fig. 2. Boltzmann viscoelastic model (uniaxial representation).

It is easy to observe, see Fig. 2, that the stress level for each part of the model, elastic and viscoelastic, is the same,

$$\sigma_{ij} = \sigma^{\mathrm{e}}_{ij} = \sigma^{\mathrm{ve}}_{ij}, \tag{13}$$

where σ_{ij}, σ^{e}_{ij}, and $\sigma^{\mathrm{ve}}_{ij}$ are, respectively, total, elastic and viscoelastic stress parts. The total strain can be decomposed into its elastic and viscoelastic parts, i.e.:

$$\varepsilon_{lm} = \varepsilon^{\mathrm{e}}_{lm} + \varepsilon^{\mathrm{ve}}_{lm}. \tag{14}$$

From Fig. 2, one may observe that the viscoelastic stress is the summation of a viscous and an elastic part, as follows:

$$\sigma^{\mathrm{ve}}_{ij} = \sigma^{\mathrm{el}}_{ij} + \sigma^{\mathrm{v}}_{ij}, \tag{15}$$

where σ^{v}_{ij} is the viscous part and $\sigma^{\mathrm{el}}_{ij}$ is the elastic part of the stress developed in the Kelvin–Voigt fragment of the Boltzmann model.

From the previous equations, one is able to define the differential constitutive relation necessary to build the desired boundary integral equations, as described in Appendix A:

$$\sigma_{qs} = \left(\frac{E_{\mathrm{ve}}E_{\mathrm{e}}}{E_{\mathrm{ve}}+E_{\mathrm{e}}}\right)\widetilde{C}^{\gamma k}_{qs}\varepsilon_{\gamma k} + \left(\frac{E_{\mathrm{e}}E_{\mathrm{ve}}}{E_{\mathrm{ve}}+E_{\mathrm{e}}}\right)\tilde{\eta}^{\gamma k}_{qs}\dot{\varepsilon}_{\gamma k} - \left(\frac{E_{\mathrm{ve}}}{E_{\mathrm{ve}}E_{\mathrm{e}}}\right)\tilde{\eta}^{ij}_{qs}\widetilde{D}^{\gamma k}_{ij}\dot{\sigma}_{\gamma k}, \tag{16}$$

where $\widetilde{C}^{lm}_{ij}$, $\tilde{\eta}^{\gamma k}_{qs}$ and $\widetilde{D}^{\gamma k}_{ij}$ are the dimensionless constitutive tensor, the dimensionless viscoelastic compliance tensor and the inverse of the dimensionless constitutive tensor, respectively (see Appendix A).

In order to write an integral statement with only boundary values it is necessary to impose the simplification $\theta_\lambda = \theta_\mu = \gamma$. In this way expression (16) turns into:

$$\sigma_{ij} = \left(\frac{E_{\mathrm{ve}}E_{\mathrm{e}}}{E_{\mathrm{ve}}+E_{\mathrm{e}}}\right)\widetilde{C}^{lm}_{ij}\varepsilon_{lm} + \left(\frac{\gamma E_{\mathrm{ve}}E_{\mathrm{e}}}{E_{\mathrm{ve}}+E_{\mathrm{e}}}\right)\widetilde{C}^{lm}_{ij}\dot{\varepsilon}_{lm} - \left(\frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{ve}}E_{\mathrm{e}}}\right)\dot{\sigma}_{ij}. \tag{17}$$

This is the general rheological differential relation for the Boltzmann model. More complicate rheological models can be introduced following similar procedures, as the one employed in Appendix A, to achieve Eq. (17).

3. Integral equations

The BEM is based on boundary integral equations. In Section 3.1, it will be developed the integral statement for Kelvin model, due to its simplicity. After that, the procedure is extended to the Boltzmann model.

3.1. Kelvin model

3.1.1. Displacement representation

The viscoelastic integral equation for boundary or internal points is obtained here using the weighting residual technique on the differential equilibrium equation (12) written in the following form:

$$\sigma_{ij,j} + b_i = 0. \tag{18}$$

The error present in Eq. (18), when an approximate solution is adopted, can be weighted by a proper function. In this work, the Kelvin fundamental solution for elastic infinite body is adopted. Eq. (18) is weighted over the analysed domain Ω, as follows:

$$\int_\Omega u^*_{ki}(\sigma_{ij,j} + b_i)\,\mathrm{d}\Omega = 0, \tag{19}$$

where u^*_{ki} is the Kelvin fundamental solution. It represents the effect of a unit concentrate load applied at a point located in an infinite domain. Applying the divergence theorem on the first term of Eq. (19), one achieves:

$$\int_\Gamma u^*_{ki}\sigma_{ij}n_j\,\mathrm{d}\Gamma - \int_\Omega u^*_{ki,j}\sigma_{ij}\,\mathrm{d}\Omega + \int_\Omega u^*_{ki}b_i\,\mathrm{d}\Omega = 0, \tag{20}$$

where Γ is the boundary of the analysed body and n_j is the outward normal vector component. Knowing that, $\sigma_{ij}n_j = p_i$ and that $u^*_{ki,j}\sigma_{ij} = \varepsilon^*_{kij}\sigma_{ij}$, where ε^*_{kij} is the strain fundamental term, Eq. (20) turns into

$$\int_\Gamma u^*_{ki}p_i\,\mathrm{d}\Gamma - \int_\Omega \varepsilon^*_{kij}\sigma_{ij}\,\mathrm{d}\Omega + \int_\Omega u^*_{ki}b_i\,\mathrm{d}\Omega = 0. \tag{21}$$

This equation is the starting point to obtain the viscoelastic integral representations. Imposing the viscoelastic relations, Eq. (9b), on Eq. (21), results

$$\int_\Gamma u^*_{ki}p_i\,\mathrm{d}\Gamma - \int_\Omega \varepsilon^*_{kij}C^{lm}_{ij}\varepsilon_{lm}\,\mathrm{d}\Omega - \int_\Omega \varepsilon^*_{kij}\gamma C^{lm}_{ij}\dot{\varepsilon}_{lm}\,\mathrm{d}\Omega + \int_\Omega u^*_{ki}b_i\,\mathrm{d}\Omega = 0. \tag{22}$$

Knowing that

$$\varepsilon^*_{kij}C^{lm}_{ij}\varepsilon_{lm} = \sigma^*_{klm}\varepsilon_{lm} = \sigma^*_{klm}u_{l,m} = \sigma^*_{kij}u_{i,j}, \tag{23}$$

$$\varepsilon^*_{kij}\gamma C^{lm}_{ij}\dot{\varepsilon}_{lm} = \gamma\sigma^*_{klm}\dot{\varepsilon}_{lm} = \gamma\sigma^*_{klm}\dot{u}_{l,m} = \gamma\sigma^*_{kij}\dot{u}_{i,j}. \tag{24}$$

Eq. (22) turns into

$$\int_\Gamma u^*_{ki}p_i\,\mathrm{d}\Gamma - \int_\Omega \sigma^*_{kij}u_{i,j}\,\mathrm{d}\Omega - \gamma\int_\Omega \sigma^*_{kij}\dot{u}_{i,j}\,\mathrm{d}\Omega + \int_\Omega u^*_{ki}b_i\,\mathrm{d}\Omega = 0. \tag{25}$$

Integrating by parts, the second and third terms of Eq. (25) one achieves

$$\int_\Gamma u^*_{ki}p_i\,\mathrm{d}\Gamma - \int_\Gamma \sigma^*_{kij}n_ju_i\,\mathrm{d}\Gamma + \int_\Omega \sigma^*_{kij,j}u_i\,\mathrm{d}\Omega - \gamma\int_\Gamma \sigma^*_{kij}n_j\dot{u}_i\,\mathrm{d}\Gamma + \gamma\int_\Omega \sigma^*_{kij,j}\dot{u}_i\,\mathrm{d}\Omega + \int_\Omega u^*_{ki}b_i\,\mathrm{d}\Omega = 0. \tag{26}$$

Eq. (26) can be rewritten by using the fundamental equilibrium equation, i.e.,

$$\sigma^*_{kij,j} = -\delta(p,s)\delta_{ki}, \tag{27}$$

where $\delta(p,s)$ is Dirac's Delta distribution, s is a field point and p is the source location. Applying Eq. (27) into Eq. (26), taking into consideration Dirac's Delta properties and that $\sigma^*_{kij}n_j = p^*_{ki}$, results

$$\overline{C}_{ki}u_i(p) + \gamma\overline{C}_{ki}\dot{u}_i(p) = \int_\Gamma u^*_{ki}p_i\,\mathrm{d}\Gamma - \int_\Gamma p^*_{ki}u_i\,\mathrm{d}\Gamma - \gamma\int_\Gamma p^*_{ki}\dot{u}_i\,\mathrm{d}\Gamma + \int_\Omega u^*_{ki}b_i\,\mathrm{d}\Omega. \tag{28}$$

The term $\overline{C}_{ki}$ is the same obtained in the elastostatic formulations and can be found in standard Boundary Element references (see e.g. Brebbia et al. (1984) and Brebbia and Dominguez (1992)). Eq. (28) is the alternative viscoelastic integral representation for the Kelvin–Voigt model. The body force domain integral can be easily transformed to its boundary representation, resulting an expression written exclusively for boundary values (see e.g. Coda and Venturini (1998)). If the body force b_i is constant, the explicit integral expression is

$$\int_\Omega u^*_{ki}b_i\,\mathrm{d}\Omega = b_i\int_\Omega u^*_{ki}\,\mathrm{d}\Omega = b_i\int_\theta\int_r u^*_{ki}r\,\mathrm{d}r\,\mathrm{d}\theta = b_i\int_\Gamma\int_r u^*_{ki}r\,\mathrm{d}r\frac{1}{r}\frac{\partial r}{\partial n}\,\mathrm{d}\Gamma = b_i\int_\Gamma B^*_{ki}\,\mathrm{d}\Gamma. \tag{29}$$

For Kelvin fundamental solution, B^*_{ki} is given by

$$B^*_{ki} = \frac{r}{16\pi G(1-v)}\left[(4v-3)\left(\ln r - \frac{1}{2}\right)\delta_{ki} + r_{,k}\, r_{,i}\right]\frac{\partial r}{\partial n}. \tag{30}$$

Rewriting Eq. (28) results

$$\overline{C}_{ki}u_i(p) + \gamma\overline{C}_{ki}\dot{u}_i(p) = \int_\Gamma u^*_{ki}p_i\,\mathrm{d}\Gamma - \int_\Gamma p^*_{ki}u_i\,\mathrm{d}\Gamma - \gamma\int_\Gamma p^*_{ki}\dot{u}_i\,\mathrm{d}\Gamma + b_i\int_\Gamma B^*_{ki}\,\mathrm{d}\Gamma. \tag{31}$$

3.1.2. Stress integral representation for internal points

To write the stress integral representation for internal points one starts by achieving the strain integral representation. At internal points the displacement integral representation is given by

$$u_k(p) + \gamma\dot{u}_k(p) = \int_\Gamma u^*_{ki}p_i\,\mathrm{d}\Gamma - \int_\Gamma p^*_{ki}u_i\,\mathrm{d}\Gamma - \gamma\int_\Gamma p^*_{ki}\dot{u}_i\,\mathrm{d}\Gamma + b_i\int_\Gamma B^*_{ki}\,\mathrm{d}\Gamma. \tag{32}$$

The kinematical relation for small strains, is adopted:

$$\varepsilon_{ke} = \tfrac{1}{2}(u_{k,e} + u_{e,k}). \tag{33}$$

Applying the above definition on Eq. (32) and considering that the derivatives are done with respect to the source point location, one finds

$$\varepsilon_{ke}(p) + \gamma\dot{\varepsilon}_{ke}(p) = \int_\Gamma \varepsilon^*_{kie}p_i\,\mathrm{d}\Gamma - \int_\Gamma \hat{p}^*_{kie}u_i\,\mathrm{d}\Gamma - \gamma\int_\Gamma \hat{p}^*_{kie}\dot{u}_i\,\mathrm{d}\Gamma + b_i\int_\Gamma \hat{B}^*_{kie}\,\mathrm{d}\Gamma. \tag{34}$$

The terms $\hat{p}^*_{kie}$, ε^*_{kie} and $\hat{B}^*_{kie}$ are defined in Appendix C. The total stress is obtained using the constitutive relation (9b) over Eq. (34), resulting

$$\sigma^{\mathrm{e}}_{\rho q}(p) + \sigma^{\mathrm{v}}_{\rho q}(p) = \int_\Gamma \bar{\sigma}^*_{\rho iq}p_i\,\mathrm{d}\Gamma - \int_\Gamma \bar{p}^*_{\rho iq}u_i\,\mathrm{d}\Gamma - \gamma\int_\Gamma \bar{p}^*_{\rho iq}\dot{u}_i\,\mathrm{d}\Gamma + b_i\int_\Gamma \overline{B}^*_{\rho iq}\,\mathrm{d}\Gamma. \tag{35}$$

Using Eq. (2), Eq. (35) turns into

$$\sigma_{\rho q}(p) = \int_\Gamma \bar{\sigma}^*_{\rho iq}p_i\,\mathrm{d}\Gamma - \int_\Gamma \bar{p}^*_{\rho iq}u_i\,\mathrm{d}\Gamma - \gamma\int_\Gamma \bar{p}^*_{\rho iq}\dot{u}_i\,\mathrm{d}\Gamma + b_i\int_\Gamma \overline{B}^*_{\rho iq}\,\mathrm{d}\Gamma, \tag{36}$$

where $\bar{\sigma}^*_{\rho iq}$, $\bar{p}^*_{\rho iq}$ and $\overline{B}^*_{\rho iq}$ are defined in Appendix C.

In order to determine the elastic and viscous stresses, from Eq. (36), one should apply a special scheme proposed by Mesquita et al. (2001). In this scheme Eq. (3) is written in the following form:

$$\dot{\sigma}^{\mathrm{el}}_{ij} = C^{lm}_{ij}\dot{\varepsilon}^{\mathrm{ve}}_{lm} = \frac{1}{\gamma}\gamma C^{lm}_{ij}\dot{\varepsilon}^{\mathrm{ve}}_{lm} = \frac{1}{\gamma}\sigma^{\mathrm{v}}_{ij} \quad\Rightarrow\quad \sigma^{\mathrm{v}}_{ij} = \gamma\dot{\sigma}^{\mathrm{el}}_{ij}. \tag{37}$$

Substituting the above relation into Eq. (2) one achieves the following time differential equation:

$$\gamma\dot{\sigma}^{\mathrm{e}}_{ij} + \sigma^{\mathrm{e}}_{ij} - \sigma_{ij} = 0. \tag{38}$$

Eq. (38) is solved numerically by adopting linear approximation for the elastic stress. This numerical procedure is shown in the next section.

When relaxation functions are adopted, one solves locally this equation by imposing constant stress over a time-step. This procedure results in the following relation: $\sigma^{\mathrm{e}}_{ij} = \sigma_{ij}(1 - \mathrm{e}^{-t/\gamma})$. This is done before writing the global equilibrium equation. In the authors' opinion, this procedure violates the continuity statement of the viscous stress field, mainly in viscoplastic applications.

3.1.3. Algebraic treatment

The kernels present in Eqs. (32) and (36) are the usual ones of ordinary static boundary element formulations. The boundary Γ of the analysed domain is divided into various boundary elements Γ_c, over which the variables are approximated, as follows:

$$
\begin{aligned}
p_i &= \phi^\alpha P_i^\alpha, \\
u_i &= \phi^\alpha U_i^\alpha, \\
\dot{u}_i &= \phi^\alpha \dot{U}_i^\alpha,
\end{aligned} \tag{39}
$$

where ϕ^α are shape functions and α is the element node. The values P_i^α, U_i^α and $\dot{U}_i^\alpha$ are nodal variables. Adopting these approximations the integral representation for displacement and stress are written as:

$$
\begin{aligned}
\overline{C}_{ki}U_i(p) + \gamma\overline{C}_{ki}\dot{U}_i(p) = {}& \sum_{c=1}^{n_c}\int_{\Gamma_c} u_{ki}^*\phi^\alpha\,\mathrm{d}\Gamma_c P_i^\alpha - \sum_{c=1}^{n_c}\int_{\Gamma_c} p_{ki}^*\phi^\alpha\,\mathrm{d}\Gamma_c U_i^\alpha - \gamma\sum_{c=1}^{n_c}\int_{\Gamma_c} p_{ki}^*\phi^\alpha\,\mathrm{d}\Gamma_c \dot{U}_i^\alpha \\
& + b_i\sum_{c=1}^{n_c}\int_{\Gamma_c} B_{ki}^*\,\mathrm{d}\Gamma_c,
\end{aligned} \tag{40}
$$

$$
\sigma_{\rho q}(p) = \sum_{c=1}^{n_c}\int_{\Gamma_c}\bar{\sigma}_{\rho iq}^*\phi^\alpha\,\mathrm{d}\Gamma_c P_i^\alpha - \sum_{c=1}^{n_c}\int_{\Gamma_c}\bar{p}_{\rho iq}^*\phi^\alpha\,\mathrm{d}\Gamma_c U_i^\alpha - \gamma\sum_{c=1}^{n_c}\int_{\Gamma_c}\bar{p}_{\rho iq}^*\phi^\alpha\,\mathrm{d}\Gamma_c \dot{U}_i^\alpha + b_i\sum_{c=1}^{n_c}\int_{\Gamma_c}\overline{B}_{\rho iq}^*\,\mathrm{d}\Gamma_c. \tag{41}
$$

After chosen the number of source points equal to the number of nodes and calculating all integrals, results

$$HU(t) + \gamma H\dot{U}(t) = GP(t) + Bb(t), \tag{42}$$

$$\sigma(t) = G'P(t) - H'U(t) - \gamma H'\dot{U}(t) + B'b(t), \tag{43}$$

where t represents time.

To solve the time differential matrix equation (42) it is necessary to approximate velocity in time. This is done adopting linear behaviour along time, as follows:

$$\dot{U}_{s+1} = \frac{U_{s+1} - U_s}{\Delta t}. \tag{44}$$

Applying Eq. (44) into (42) the following linear time marching process is achieved:

$$\overline{H}U_{s+1} = GP_{s+1} + F_s, \tag{45}$$

where

$$\overline{H} = \left(1 + \frac{\gamma}{\Delta t}\right)H, \tag{46}$$

$$F_s = \frac{\gamma}{\Delta t}HU_s + Bb_{s+1}. \tag{47}$$

As past values are known, it is necessary only to solve the system (45) for actual time, i.e., t_{s+1}, and go to the next time-step. The boundary conditions along time are prescribed by interchanging columns of $\overline{H}$ and G.

In order to calculate the total stress one applies Eqs. (44) and (43), written for the instant t_{s+1}, as follows:

$$\sigma_{s+1} = G'P_{s+1} - H'U_{s+1} - \gamma H'\dot{U}_{s+1} + B'b_{s+1}. \tag{48}$$

Assuming the same approximation for elastic stress as the one adopted for velocity, i.e.:

$$\dot{\sigma}^{\mathrm{e}}_{s+1} = \frac{\sigma^{\mathrm{e}}_{s+1} - \sigma^{\mathrm{e}}_{s}}{\Delta t} \tag{49}$$

and substituting Eq. (49) into (38) results

$$\sigma^{\mathrm{e}}_{s+1} = \left(\sigma_{s+1} + \frac{\gamma}{\Delta t}\sigma^{\mathrm{e}}_{s}\right) \Big/ \left(1 + \frac{\gamma}{\Delta t}\right). \tag{50}$$

As σ^{e}_{s} is known and σ_{s+1} is obtained by Eq. (48), the elastic stress holds from Eq. (50) and the viscous part comes from Eq. (2).

3.2. Boltzmann model

In this section, the viscoelastic formulation developed previously for the Kelvin model is extended to include the Boltzmann viscoelastic relation.

Applying the weighting residual technique over the differential equilibrium equation (18) the following integral equation is achieved:

$$\int_\Gamma u^*_{ki} p_i \,\mathrm{d}\Gamma - \int_\Omega \varepsilon^*_{kij}\sigma_{ij}\,\mathrm{d}\Omega + \int_\Omega u^*_{ki} b_i \,\mathrm{d}\Omega = 0. \tag{51}$$

This is the same equation used in the previous sections. Imposing on Eq. (51) the rheological relation (17) one achieves:

$$\int_\Gamma u^*_{ki} p_i \,\mathrm{d}\Gamma - \frac{E_{\mathrm{e}}E_{\mathrm{ve}}}{E_{\mathrm{e}}+E_{\mathrm{ve}}}\int_\Omega \varepsilon^*_{kij} C^{lm}_{ij}\varepsilon_{lm}\,\mathrm{d}\Omega - \frac{\gamma E_{\mathrm{e}}E_{\mathrm{ve}}}{E_{\mathrm{e}}+E_{\mathrm{ve}}}\int_\Omega \varepsilon^*_{kij} C^{lm}_{ij}\dot{\varepsilon}_{lm}\,\mathrm{d}\Omega + \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}}+E_{\mathrm{ve}}}\int_\Omega \varepsilon^*_{kij}\dot{\sigma}_{ij}\,\mathrm{d}\Omega + \int_\Omega u^*_{ki} b_i\,\mathrm{d}\Omega = 0. \tag{52}$$

Following the steps described in Appendix B one achieves the displacement integral equation for Boltzmann model, as

$$\overline{C}_{ki}u_i(p) = \frac{E_{\mathrm{e}}+E_{\mathrm{ve}}}{E_{\mathrm{ve}}}\int_\Gamma u^*_{ki}p_i\,\mathrm{d}\Gamma - \int_\Gamma p^*_{ki}u_i\,\mathrm{d}\Gamma - \gamma\int_\Gamma p^*_{ki}\dot{u}_i\,\mathrm{d}\Gamma - \gamma\overline{C}_{ki}\dot{u}_i(p) + \gamma\left[\int_\Gamma u^*_{ki}\dot{p}_i\,\mathrm{d}\Gamma + \dot{b}_i\int_\Gamma B^*_{ki}\,\mathrm{d}\Gamma\right] + \frac{E_{\mathrm{e}}+E_{\mathrm{ve}}}{E_{\mathrm{ve}}}b_i\int_\Gamma B^*_{ki}\,\mathrm{d}\Gamma. \tag{53}$$

The stress integral representation for total stress following the Boltzmann viscoelastic model is given by

$$\sigma_{\rho q}(p) = \int_\Gamma \bar{\sigma}^*_{\rho iq}p_i\,\mathrm{d}\Gamma - \frac{E_{\mathrm{ve}}}{E_{\mathrm{e}}+E_{\mathrm{ve}}}\int_\Gamma \bar{p}^*_{\rho iq}u_i\,\mathrm{d}\Gamma - \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}}+E_{\mathrm{ve}}}\int_\Gamma \bar{p}^*_{\rho iq}\dot{u}_i\,\mathrm{d}\Gamma + \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}}+E_{\mathrm{ve}}}\left[\int_\Gamma \bar{\sigma}^*_{\rho iq}\dot{p}_i\,\mathrm{d}\Gamma + \dot{b}_i\int_\Gamma \overline{B}^*_{\rho iq}\,\mathrm{d}\Gamma\right] + b_i\int_\Gamma \overline{B}^*_{\rho iq}\,\mathrm{d}\Gamma - \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}}+E_{\mathrm{ve}}}\dot{\sigma}_{\rho q}(p). \tag{54}$$

The functions $\bar{p}^*_{\rho iq}$, $\bar{\sigma}^*_{\rho iq}$ and $\overline{B}^*_{\rho iq}$ were defined previously for the Kelvin model. The elastic instantaneous stress and the viscoelastic stress are equal to the total stress, given by Eq. (54), see Eqs. (13) and (15). It is necessary to separate the viscous and elastic parts of the viscoelastic stress. This is done by Eq. (A.7) in its time differential form, as:

$$\dot{\sigma}^{\mathrm{el}}_{ij} = C^{lm}_{ij}\dot{\varepsilon}^{\mathrm{ve}}_{lm} = \frac{1}{\gamma}\gamma C^{lm}_{ij}\dot{\varepsilon}^{\mathrm{ve}}_{lm} = \frac{1}{\gamma}\sigma^{\mathrm{v}}_{ij} \quad\Rightarrow\quad \sigma^{\mathrm{v}}_{ij} = \gamma\dot{\sigma}^{\mathrm{el}}_{ij}. \tag{55}$$

Substituting Eq. (55) into Eq. (15) results following differential equation:

$$\gamma\dot{\sigma}^{\mathrm{el}}_{ij} + \sigma^{\mathrm{el}}_{ij} - \sigma_{ij} = 0. \tag{56}$$

This equation is similar to Eq. (38) written for the Kelvin model, but is understood now in the Boltzmann model sense. This equation is solved numerically by adopting a proper time approximation for the elastic stress rate.

3.2.1. Algebraic treatment

The kernels to be integrated in Eqs. (53) and (54) are the same usually found in elastic boundary elements formulations. Adopting the same approximations used for the Kelvin model and including the following approximation for surface force rate,

$$\dot{p}_i = \phi^\alpha \dot{P}^\alpha_i \tag{57}$$

one writes

$$\begin{aligned}\overline{C}_{ki}U_i(p) = {} & \frac{E_{\mathrm{e}} + E_{\mathrm{ve}}}{E_{\mathrm{ve}}}\sum_{c=1}^{n_c}\int_{\Gamma_c} u^*_{ki}\phi^\alpha\,\mathrm{d}\Gamma_c P^\alpha_i - \sum_{c=1}^{n_c}\int_{\Gamma_c} p^*_{ki}\phi^\alpha\,\mathrm{d}\Gamma_c U^\alpha_i - \gamma\sum_{c=1}^{n_c}\int_{\Gamma_c} p^*_{ki}\phi^\alpha\,\mathrm{d}\Gamma_c \dot{U}^\alpha_i - \gamma\overline{C}_{ki}\dot{U}_i(p) \\ & + \gamma\sum_{c=1}^{n_c}\int_{\Gamma_c} u^*_{ki}\phi^\alpha\,\mathrm{d}\Gamma_c\dot{P}^\alpha_i + \gamma\sum_{c=1}^{n_c}\int_{\Gamma_c} B^*_{ki}\,\mathrm{d}\Gamma_c\dot{b}_i + \frac{E_{\mathrm{e}} + E_{\mathrm{ve}}}{E_{\mathrm{ve}}}\sum_{c=1}^{n_c}\int_{\Gamma_c} B^*_{ki}\,\mathrm{d}\Gamma_c b_i,\end{aligned} \tag{58}$$

$$\begin{aligned}\sigma_{\rho q}(p) = {} & \sum_{c=1}^{n_c}\int_{\Gamma_c}\bar{\sigma}^*_{\rho iq}\phi^\alpha\,\mathrm{d}\Gamma_c P^\alpha_i - \frac{E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}}\sum_{c=1}^{n_c}\int_{\Gamma_c}\bar{p}^*_{\rho iq}\phi^\alpha\,\mathrm{d}\Gamma_c U^\alpha_i - \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}}\sum_{c=1}^{n_c}\int_{\Gamma_c}\bar{p}^*_{\rho iq}\phi^\alpha\,\mathrm{d}\Gamma_c\dot{U}^\alpha_i \\ & + \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}}\sum_{c=1}^{n_c}\int_{\Gamma_c}\bar{\sigma}^*_{\rho iq}\phi^\alpha\,\mathrm{d}\Gamma_c\dot{P}^\alpha_i + \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}}\sum_{c=1}^{n_c}\int_{\Gamma_c}\overline{B}^*_{\rho iq}\,\mathrm{d}\Gamma_c\dot{b}_i + \sum_{c=1}^{n_c}\int_{\Gamma_c}\overline{B}^*_{\rho iq}\,\mathrm{d}\Gamma_c b_i \\ & - \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}}\dot{\sigma}_{\rho q}(p).\end{aligned} \tag{59}$$

After adopting the same number of source points as the nodal ones and performing all spatial integration results:

$$HU(t) = \frac{E_{\mathrm{e}} + E_{\mathrm{ve}}}{E_{\mathrm{ve}}}GP(t) - \gamma H\dot{U}(t) + \gamma G\dot{P}(t) + \gamma B\dot{b}(t) + \frac{E_{\mathrm{e}} + E_{\mathrm{ve}}}{E_{\mathrm{ve}}}Bb(t), \tag{60}$$

$$\begin{aligned}\sigma(t) = {} & G'P(t) - \frac{E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}}H'U(t) - \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}}H'\dot{U}(t) + \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}}G'\dot{P}(t) + \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}}B'\dot{b}(t) + B'b(t) \\ & - \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}}\dot{\sigma}(t),\end{aligned} \tag{61}$$

where t represents time.

In order to solve the matrix time differential equation (60) the following time approximations are adopted:

$$\begin{aligned}
\dot{U}_{s+1} &= \frac{U_{s+1} - U_s}{\Delta t}, \\
\dot{P}_{s+1} &= \frac{P_{s+1} - P_s}{\Delta t}, \\
\dot{b}_{s+1} &= \frac{b_{s+1} - b_s}{\Delta t}, \\
\dot{\sigma}_{s+1} &= \frac{\sigma_{s+1} - \sigma_s}{\Delta t}, \\
\dot{\sigma}^{\mathrm{el}}_{s+1} &= \frac{\sigma^{\mathrm{el}}_{s+1} - \sigma^{\mathrm{el}}_s}{\Delta t}.
\end{aligned} \tag{62}$$

Introducing the first three approximations of Eq. (62) into Eq. (61), it turns into a time marching process,

$$\overline{H} U_{s+1} = \left(\frac{\gamma}{\Delta t} + \frac{E_{\mathrm{e}} + E_{\mathrm{ve}}}{E_{\mathrm{ve}}}\right) G P_{s+1} + F_s, \tag{63}$$

where

$$\overline{H} = \left(1 + \frac{\gamma}{\Delta t}\right) H, \tag{64}$$

$$F_s = \frac{\gamma}{\Delta t} H U_s - \frac{\gamma}{\Delta t} G P_s + B\left[\left(\frac{\gamma}{\Delta t} + \frac{E_{\mathrm{e}} + E_{\mathrm{ve}}}{E_{\mathrm{ve}}}\right) b_{s+1} - \frac{\gamma}{\Delta t} b_s\right]. \tag{65}$$

The time-dependent boundary conditions are prescribed by interchanging columns of $\overline{H}$ and G. The system (63) is solved for the present instant and the results turns into past for the next time-step.

Using the results U_{s+1} and P_{s+1}, one is able to calculate $\dot{P}_{s+1}$, $\dot{U}_{s+1}$ and $\dot{b}_{s+1}$ following expressions (62). From these values, it is easy to calculate the total stress level at t_{s+1} as follows:

$$\begin{aligned}
\sigma_{s+1} = \Bigg(& G' P_{s+1} - \frac{E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}} H' U_{s+1} - \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}} H' \dot{U}_{s+1} + \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}} G' \dot{P}_{s+1} + \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}} B' \dot{b}_{s+1} \\
& + B' b_{s+1} + \frac{\gamma}{\Delta t} \frac{E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}} \sigma_s \Bigg) \Bigg/ \left(1 + \frac{\gamma}{\Delta t} \frac{E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}}\right).
\end{aligned} \tag{66}$$

The elastic stress at the viscoelastic part of the Boltzmann model $\dot{\sigma}^{\mathrm{el}}_{s+1}$ is achieved by imposing the approximation shown in Eq. (62) on the differential equation (56), as follows:

$$\sigma^{\mathrm{el}}_{s+1} = \left(\sigma_{s+1} + \frac{\gamma}{\Delta t} \sigma^{\mathrm{el}}_s\right) \Big/ \left(1 + \frac{\gamma}{\Delta t}\right). \tag{67}$$

From expression (15) and the elastic part of the viscoelastic stress, Eq. (67), one achieves the viscous stress σ^{v}_{ij}, completing the procedure.

4. Examples

4.1. Kelvin viscoelastic model

4.1.1. Simple stressed bar

This is a benchmark example, very often used to validate viscoelastic formulations. A simple bar is subjected to a longitudinal distributed load, Fig. 3. The geometry, load and physical properties are also shown in Fig. 3.

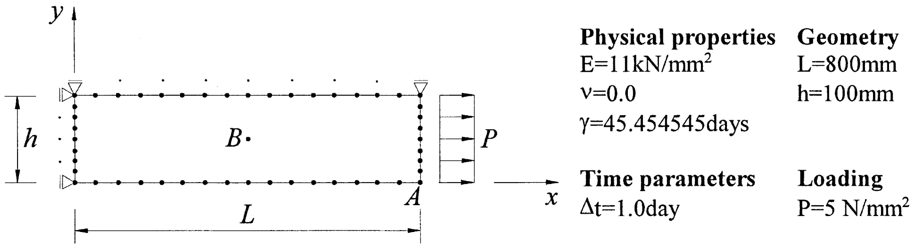

Fig. 3. Geometry, discretization and physical properties.

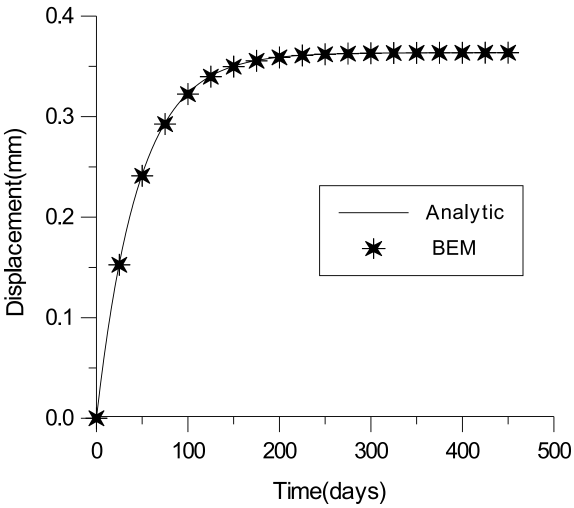

Fig. 4. Longitudinal displacement of point A.

The longitudinal displacement of point A is depicted in Fig. 4, and is compared with the analytical answer ($\nu = 0$). As it can be seen, the results are practically the same.

Fig. 5 shows the numerical elastic, viscous and total stresses, σ^{e}_{11}, σ^{v}_{11}, σ_{11}, at point B. The results are compared with the analytical ones. Again, the formulation behaviour is almost perfect.

4.1.2. Thick cylinder subjected to internal pressure

In this example the behaviour of a thick cylinder subjected to an internal pressure P is analysed. Due to the double symmetry only a quarter of the structure is discretized, see Fig. 6. The geometry and physical properties are depicted in Fig. 6.

The inner and outer wall radial displacements obtained applying this numerical formulation are compared with the analytical ones in Figs. 7 and 8, respectively. The numerical results were obtained adopting a time-step of one day.

As for the first example the numerical behaviour are almost the same as the analytical ones. Information about stability, general loading and infinite domains applications are given in examples related to the Boltzmann model.

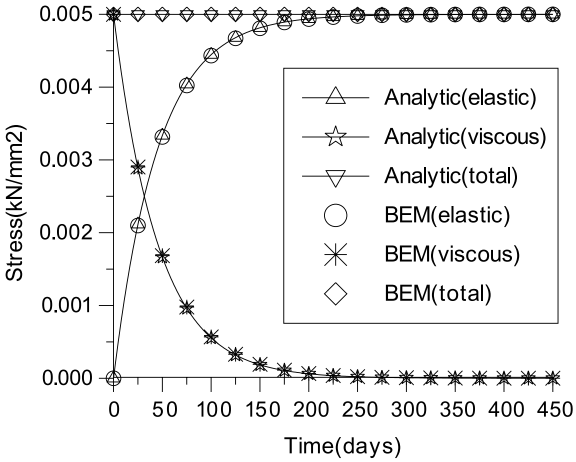

Fig. 5. Stresses at point B.

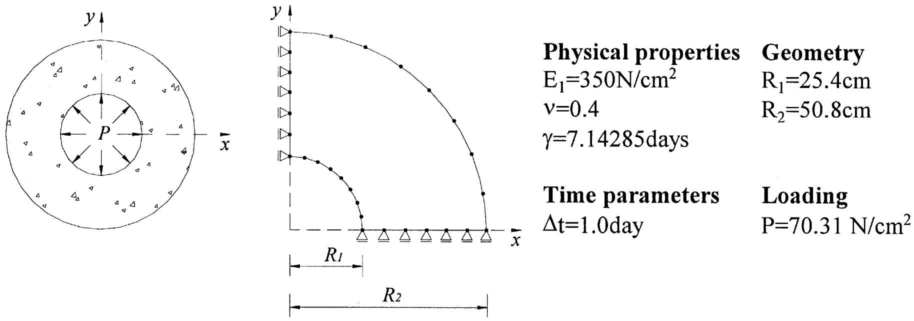

Fig. 6. Geometry, discretization and physical properties.

4.2. Boltzmann model

4.2.1. Simple stressed bar

In order to verify the behaviour of the Boltzmann numerical formulation, a simple stressed bar is analysed. The structure is modelled by adopting quadratic boundary elements, a quarter of the body is discretized. The geometry, discretization and physical properties are given in Fig. 9.

The longitudinal displacement behaviour of point A is given in Fig. 10. It compares very well with the analytical solution ($\nu = 0$).

In Fig. 11 the stability of the method, regarding time-steps length, is shown. One can see that the results are very stable. The time-step length varied from 1 to 5 days. The analysis has a total duration of 405 days.

Fig. 12 shows the stresses components $\sigma_{11}^{\mathrm{el}}$ (elastic at viscoelastic part), σ_{11}^{v} (viscous) and σ_{11} (total) at point B.

As for displacement, the stresses numerical behaviour is almost the same as the analytical one. It is important to note that no superposition is used to run the Boltzmann model examples, the “jump”, in displacement at the first time-step is the numerical solution.

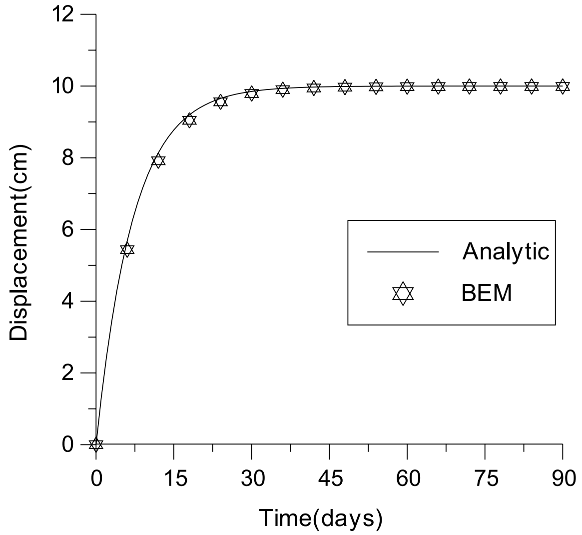

Fig. 7. Inner wall radial displacement.

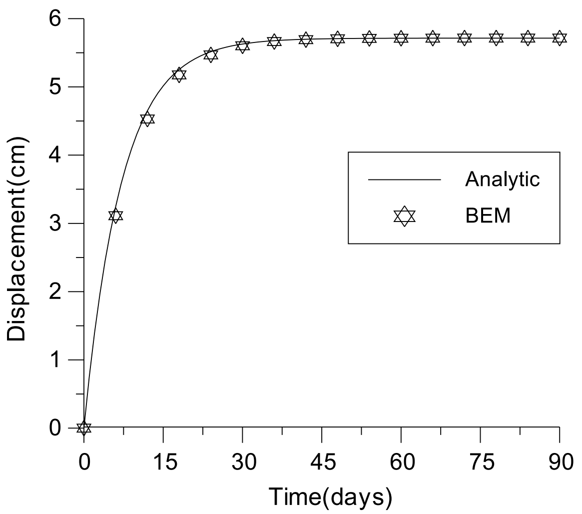

Fig. 8. Outer wall radial displacement.

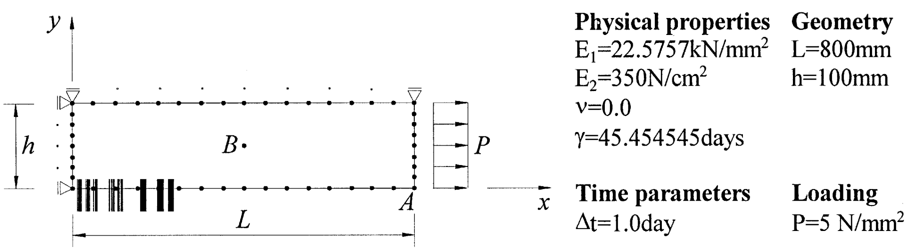

Fig. 9. Geometry, discretization and physical properties.

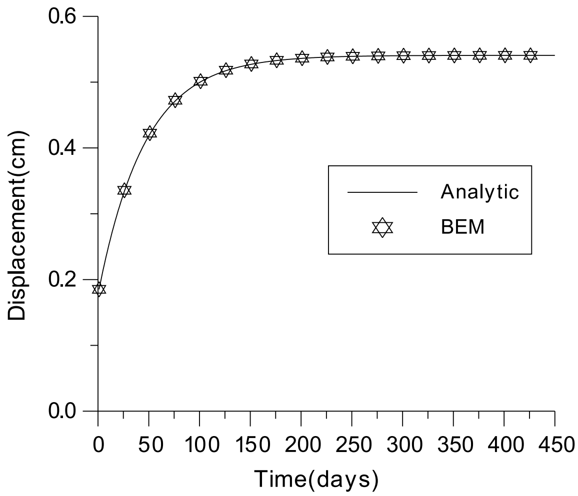

Fig. 10. Longitudinal displacement at point A.

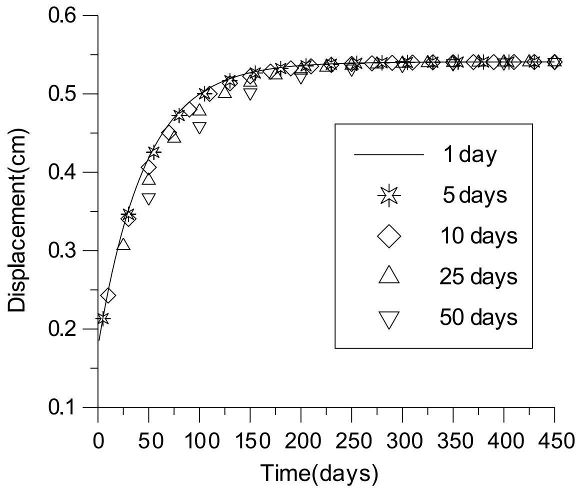

Fig. 11. Time step dependence (stability).

In Fig. 13 the displacement of point A is shown when the load is removed at day 200. The adopted time-step is one day.

4.2.2. Tunnel subjected to internal pressure

A circular cavity is modelled by the proposed viscoelastic formulation. This example exhibits that using the proposed formulation it is possible solving an exterior Boltzmann (or general) viscoelastic problem using only a boundary discretization. A constant internal pressure P is applied over the interior surface of the cavity. The cavity is modelled by adopting quadratic boundary elements, as depicted in Fig. 14. The viscoelastic properties, discretization and geometry are shown in Fig. 14.

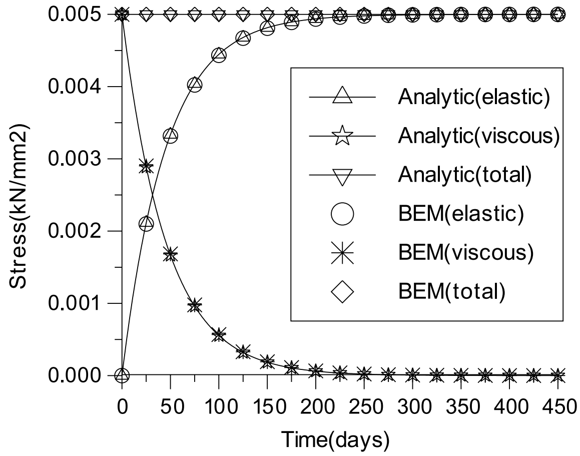

Fig. 12. Elastic stress σ_{11}^{el}, viscous σ_{11}^{v} and total σ_{11} at point *B*.

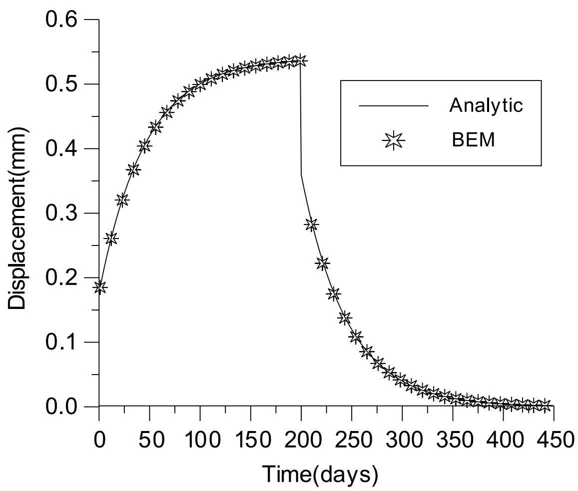

Fig. 13. Displacement behaviour of point A, load removed at day 200.

The tunnel radial displacement behaviour is shown in Fig. 15. The numerical result is compared with the analytical one, considering plane stress assumptions.

Looking at Fig. 15, it is hard distinguishing the numerical and the analytical results, what makes clear the accuracy of the proposed methodology.

5. Conclusions

It has been shown, along the paper, a new way to perform viscoelastic analysis by the BEM. It consists in considering the viscous elastic relation as a non-local property of the continuum. From this assumption, the time integration should be done after spatial approximation. Following a weighting residual procedure and a proper kinematical relation between strain velocity and material velocities of neighbour points it is

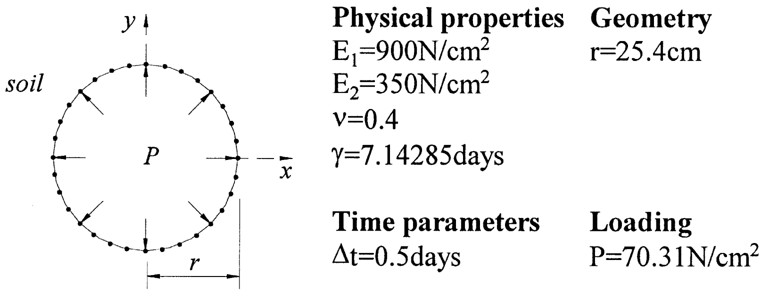

Fig. 14. Geometry, discretization and physical properties.

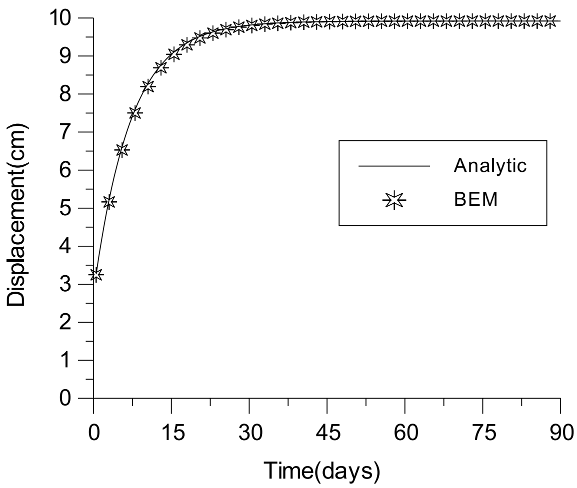

Fig. 15. Inner wall radial displacement.

possible to write the boundary integral representation for displacement and velocity. The main advantage of the presented technique is that the integral representation posses only boundary values. It has been imposed a spatial approximation for boundary values achieving a system of time differential equations. This system is easily solved by adopting linear time approximation for velocity and boundary traction rates.

A very elegant treatment is given for the stress determination. No domain approximations were assumed and only integral equations were applied. Linear time approximation was adopted for stress rates. The proposed formulation has been developed and implemented for both Kelvin and Boltzmann viscoelastic models, showing that any desired viscous model could be added to the formulation following similar steps. Four examples are shown in order to demonstrate the accuracy, stability and generality of the technique.

Appendix A. Boltzmann viscoelastic differential relation

In this appendix some necessary steps to achieve expressions (16) and (17) from Eqs. (13) to (15) are presented. It is important to show how to manage constitutive relations to achieve expressions that are more general.

The elastic (instantaneous) part of the Boltzmann viscoelastic model is governed by the Hooke's Law, i.e.:

$$\sigma_{ij}^{\mathrm{e}} = C_{ij}^{lm}\varepsilon_{lm}^{\mathrm{e}} = E_{\mathrm{e}}\widetilde{C}_{ij}^{lm}\varepsilon_{lm}^{\mathrm{e}}, \tag{A.1}$$

where C_{ij}^{lm} is the usual elastic constitutive tensor, given by

$$C_{ij}^{lm} = \lambda\delta_{ij}\delta_{lm} + \mu(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) \tag{A.2}$$

in which λ and μ are Lamè constants, given in terms of Poisson's ratio and Young's modulus as:

$$\lambda = E_{\mathrm{e}}\frac{v}{(1+v)(1-2v)} = E_{\mathrm{e}}\tilde{\lambda}, \tag{A.3}$$

$$\mu = E_{\mathrm{e}}\frac{1}{2(1+v)} = E_{\mathrm{e}}\tilde{\mu}, \tag{A.4}$$

$$\widetilde{C}_{ij}^{lm} = \tilde{\lambda}\delta_{ij}\delta_{lm} + \tilde{\mu}(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) = C_{ij}^{lm}/E_{\mathrm{e}}, \tag{A.5}$$

where, for convenience, we defined new dimensionless values $\tilde{\lambda}$, $\tilde{\mu}$ and $\widetilde{C}_{ij}^{lm}$.

Using the same idea, one writes the elastic part of the Kelvin viscoelastic stress as follows:

$$\sigma_{ij}^{\mathrm{el}} = C_{ij}^{lm}\varepsilon_{lm}^{\mathrm{ve}} = E_{\mathrm{ve}}\widetilde{C}_{ij}^{lm}\varepsilon_{lm}^{\mathrm{ve}}. \tag{A.6}$$

The viscous stress is written in a general form as:

$$\sigma_{ij}^{\mathrm{v}} = \eta_{ij}^{lm}\dot{\varepsilon}_{lm}^{\mathrm{ve}} = E_{\mathrm{ve}}\tilde{\eta}_{ij}^{lm}\dot{\varepsilon}_{lm}^{\mathrm{ve}} = E_{\mathrm{ve}}\gamma\widetilde{C}_{ij}^{lm}\dot{\varepsilon}_{lm}^{\mathrm{ve}}. \tag{A.7}$$

The viscous stress/strain rate relation is given by the tensor η_{ij}^{lm}, written as:

$$\tilde{\eta}_{ij}^{lm} = \theta_{\lambda}\tilde{\lambda}\delta_{ij}\delta_{lm} + \theta_{\mu}\tilde{\mu}(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) = \eta_{ij}^{lm}/E_{\mathrm{ve}}, \tag{A.8}$$

where, as for the Kelvin model, $\theta_{\lambda} = \theta_{\mu}$ are the viscous parameters.

The inverse representation of Eq. (A.5) is

$$\widetilde{D}_{ij}^{lm} = (\widetilde{C}_{ij}^{lm})^{-1} = \frac{1}{2}\left[\frac{1}{2}(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) - \frac{1}{1+v}\delta_{ij}\delta_{lm}\right]. \tag{A.9}$$

Applying Eq. (A.9) in Eqs. (A.1) and (A.6) and taking into account Eq. (13) one writes:

$$\varepsilon_{lm}^{\mathrm{e}} = \frac{1}{E_{\mathrm{e}}}\widetilde{D}_{lm}^{ij}\sigma_{ij}, \tag{A.10}$$

$$\varepsilon_{lm}^{\mathrm{ve}} = \frac{1}{E_{\mathrm{ve}}}\widetilde{D}_{lm}^{ij}\sigma_{ij}^{\mathrm{el}}. \tag{A.11}$$

Form Eq. (15) one transforms Eq. (A.11) into

$$\varepsilon_{lm}^{\mathrm{ve}} = \frac{1}{E_{\mathrm{ve}}}\widetilde{D}_{lm}^{ij}(\sigma_{ij}^{\mathrm{ve}} - \sigma_{ij}^{\mathrm{v}}) = \frac{1}{E_{\mathrm{ve}}}\widetilde{D}_{lm}^{ij}(\sigma_{ij} - \sigma_{ij}^{\mathrm{v}}). \tag{A.12}$$

Assuming relation (A.7) one writes Eq. (A.12) as follows:

$$\varepsilon_{lm}^{\mathrm{ve}} = \frac{1}{E_{\mathrm{ve}}}\widetilde{D}_{lm}^{ij}\sigma_{ij} - \widetilde{D}_{lm}^{ij}\tilde{\eta}_{ij}^{\gamma k}\dot{\varepsilon}_{\gamma k}^{\mathrm{ve}}. \tag{A.13}$$

Substituting expressions (14) and (A.10) into Eq. (A.13) and rearranging terms, results:

$$\varepsilon_{lm} = \widetilde{D}_{lm}^{ij}\left(\frac{1}{E_{\mathrm{ve}}} + \frac{1}{E_{\mathrm{e}}}\right)\sigma_{ij} - \widetilde{D}_{lm}^{ij}\tilde{\eta}_{ij}^{\gamma k}\dot{\varepsilon}_{\gamma k}^{\mathrm{ve}}. \tag{A.14}$$

Multiplying Eq. (A.14) by C_{qs}^{lm}, given in Eq. (A.5), results:

$$\left(\frac{E_{\mathrm{ve}} + E_{\mathrm{e}}}{E_{\mathrm{ve}}E_{\mathrm{e}}}\right)\sigma_{qs} = \widetilde{C}_{qs}^{\gamma k}\varepsilon_{\gamma k} + \tilde{\eta}_{qs}^{\gamma k}\dot{\varepsilon}_{\gamma k}^{\mathrm{ve}}. \tag{A.15}$$

Differentiating Eq. (14) with respect to time, one writes:

$$\dot{\varepsilon}_{\gamma k}^{\mathrm{ve}} = \dot{\varepsilon}_{\gamma k} - \dot{\varepsilon}_{\gamma k}^{\mathrm{e}}. \tag{A.16}$$

Substituting relation (A.16) into expression (A.15) and rearranging indices, results:

$$\left(\frac{E_{\mathrm{ve}} + E_{\mathrm{e}}}{E_{\mathrm{ve}}E_{\mathrm{e}}}\right)\sigma_{qs} = \widetilde{C}_{qs}^{\gamma k}\varepsilon_{\gamma k} + \tilde{\eta}_{qs}^{\gamma k}\dot{\varepsilon}_{\gamma k} - \tilde{\eta}_{qs}^{\gamma k}\dot{\varepsilon}_{\gamma k}^{\mathrm{e}}. \tag{A.17}$$

Differentiating, with respect to time, Eq. (A.10) and substituting into Eq. (A.17) one achieves:

$$\sigma_{qs} = \left(\frac{E_{\mathrm{ve}}E_{\mathrm{e}}}{E_{\mathrm{ve}} + E_{\mathrm{e}}}\right)\widetilde{C}_{qs}^{\gamma k}\varepsilon_{\gamma k} + \left(\frac{E_{\mathrm{e}}E_{\mathrm{ve}}}{E_{\mathrm{ve}} + E_{\mathrm{e}}}\right)\tilde{\eta}_{qs}^{\gamma k}\dot{\varepsilon}_{\gamma k} - \left(\frac{E_{\mathrm{ve}}}{E_{\mathrm{ve}}E_{\mathrm{e}}}\right)\tilde{\eta}_{qs}^{ij}\widetilde{D}_{ij}^{\gamma k}\dot{\sigma}_{\gamma k}. \tag{A.18}$$

In order to write an integral statement with only boundary values it is necessary to impose the simplification $\theta_\lambda = \theta_\mu = \gamma$. In this way expression (A.18) turns into:

$$\sigma_{ij} = \left(\frac{E_{\mathrm{ve}}E_{\mathrm{e}}}{E_{\mathrm{ve}} + E_{\mathrm{e}}}\right)\widetilde{C}_{ij}^{lm}\varepsilon_{lm} + \left(\frac{\gamma E_{\mathrm{ve}}E_{\mathrm{e}}}{E_{\mathrm{ve}} + E_{\mathrm{e}}}\right)\widetilde{C}_{ij}^{lm}\dot{\varepsilon}_{lm} - \left(\frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{ve}}E_{\mathrm{e}}}\right)\dot{\sigma}_{ij}. \tag{A.19}$$

This is the general rheological differential relation for the Boltzmann model. It is employed to derive the boundary element procedure proposed here. More complicate rheological models can be introduced in the following similar procedures as the employed to achieve Eq. (A.19).

Appendix B. Steps used to find Boltzmann integral representation

The following steps are important to indicate the way one can follow to write more complete viscoelastic Boundary Element formulations.

B.1. Displacement integral representation

Using the following relations inside Eq. (52),

$$\varepsilon_{kij}^{*}E_{\mathrm{e}}C_{ij}^{lm}\varepsilon_{lm} = \sigma_{klm}^{*}\varepsilon_{lm} = \sigma_{klm}^{*}u_{l,m} = \sigma_{kij}^{*}u_{i,j}, \tag{B.1}$$

$$\varepsilon_{kij}^{*}E_{\mathrm{e}}C_{ij}^{lm}\dot{\varepsilon}_{lm} = \sigma_{klm}^{*}\dot{\varepsilon}_{lm} = \sigma_{klm}^{*}\dot{u}_{l,m} = \sigma_{kij}^{*}\dot{u}_{i,j}, \tag{B.2}$$

$$\varepsilon_{kij}^{*}\dot{\sigma}_{ij} = u_{ki,j}^{*}\dot{\sigma}_{ij} \tag{B.3}$$

results:

$$\int_\Gamma u_{ki}^{*}p_i\,\mathrm{d}\Gamma - \frac{E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}}\int_\Omega \sigma_{kij}^{*}u_{i,j}\,\mathrm{d}\Omega - \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}}\int_\Omega \sigma_{kij}^{*}\dot{u}_{i,j}\,\mathrm{d}\Omega + \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}}\int_\Omega u_{ki,j}^{*}\dot{\sigma}_{ij}\,\mathrm{d}\Omega + \int_\Omega u_{ki}^{*}b_i\,\mathrm{d}\Omega = 0. \tag{B.4}$$

Integrating by parts the second, third and fourth integrals of Eq. (B.4) one finds:

$$\int_\Gamma u^*_{ki}p_i\,d\Gamma - \frac{E_{ve}}{E_e+E_{ve}}\left[\int_\Gamma \sigma^*_{kij}n_ju_i\,d\Gamma - \int_\Omega \sigma^*_{kij,j}u_i\,d\Omega\right] - \frac{\gamma E_{ve}}{E_e+E_{ve}}\left[\int_\Gamma \sigma^*_{kij}n_j\dot{u}_i\,d\Gamma - \int_\Omega \sigma^*_{kij,j}\dot{u}_i\,d\Omega\right] + \frac{\gamma E_{ve}}{E_e+E_{ve}}\left[\int_\Gamma u^*_{ki}\dot{\sigma}_{ij}n_j\,d\Gamma - \int_\Omega u^*_{ki}\dot{\sigma}_{ij,j}\,d\Omega\right] + \int_\Omega u^*_{ki}b_i\,d\Omega = 0. \tag{B.5}$$

The equilibrium equation of the analysed problem can be written as:

$$\dot{\sigma}_{ij,j} = -\dot{b}_i. \tag{B.6}$$

Using Eqs. (27) and (B.6), one writes Eq. (B.5) in the following form:

$$\overline{C}_{ki}u_i(p) = \frac{E_e+E_{ve}}{E_{ve}}\int_\Gamma u^*_{ki}p_i\,d\Gamma - \int_\Gamma p^*_{ki}u_i\,d\Gamma - \gamma\int_\Gamma p^*_{ki}\dot{u}_i\,d\Gamma - \gamma\overline{C}_{ki}\dot{u}_i(p) + \gamma\left[\int_\Gamma u^*_{ki}\dot{p}_i\,d\Gamma + \int_\Omega u^*_{ki}\dot{b}_i\,d\Omega\right] + \frac{E_e+E_{ve}}{E_{ve}}\int_\Omega u^*_{ki}b_i\,d\Omega. \tag{B.7}$$

The term $\overline{C}_{ki}$ is the same present in Eq. (28) for the Kelvin–Voigt model. Eq. (B.7) is the integral equation for displacements adopting the Boltzmann model. The last integral can be transformed to the boundary, resulting:

$$\overline{C}_{ki}u_i(p) = \frac{E_e+E_{ve}}{E_{ve}}\int_\Gamma u^*_{ki}p_i\,d\Gamma - \int_\Gamma p^*_{ki}u_i\,d\Gamma - \gamma\int_\Gamma p^*_{ki}\dot{u}_i\,d\Gamma - \gamma\overline{C}_{ki}\dot{u}_i(p) + \gamma\left[\int_\Gamma u^*_{ki}\dot{p}_i\,d\Gamma + \dot{b}_i\int_\Gamma B^*_{ki}\,d\Gamma\right] + \frac{E_e+E_{ve}}{E_{ve}}b_i\int_\Gamma B^*_{ki}\,d\Gamma. \tag{B.8}$$

B.2. Stress representation for internal points

For internal points the displacement integral representation is given by:

$$u_k(p) = \frac{E_e+E_{ve}}{E_{ve}}\int_\Gamma u^*_{ki}p_i\,d\Gamma - \int_\Gamma p^*_{ki}u_i\,d\Gamma - \gamma\int_\Gamma p^*_{ki}\dot{u}_i\,d\Gamma - \gamma\dot{u}_k(p) + \gamma\left[\int_\Gamma u^*_{ki}\dot{p}_i\,d\Gamma + \dot{b}_i\int_\Gamma B^*_{ki}\,d\Gamma\right] + \frac{E_e+E_{ve}}{E_{ve}}b_i\int_\Gamma B^*_{ki}\,d\Gamma. \tag{B.9}$$

Applying the kinematical relation (33) over Eq. (B.9) results:

$$\varepsilon_{ke}(p) = \frac{E_e+E_{ve}}{E_{ve}}\int_\Gamma \varepsilon^*_{kie}p_i\,d\Gamma - \int_\Gamma \hat{p}^*_{kie}u_i\,d\Gamma - \gamma\int_\Gamma \hat{p}^*_{kie}\dot{u}_i\,d\Gamma - \gamma\dot{\varepsilon}_{ke}(p) + \gamma\left[\int_\Gamma \varepsilon^*_{kie}\dot{p}_i\,d\Gamma + \dot{b}_i\int_\Gamma \hat{B}^*_{kie}\,d\Gamma\right] + \frac{E_e+E_{ve}}{E_{ve}}b_i\int_\Gamma \hat{B}^*_{kie}\,d\Gamma. \tag{B.10}$$

Remembering that the derivatives have been made with respect to the source point position, the values $\hat{p}^*_{kie}$, ε^*_{kie} and $\hat{B}^*_{kie}$ are given in Appendix C. The total stress is obtained by applying the constitutive relation (A.19) on the total strain, Eq. (B.10), resulting:

$$\sigma_{\rho q}(p) = \int_\Gamma \bar{\sigma}^*_{\rho iq}p_i\,d\Gamma - \frac{E_{ve}}{E_e+E_{ve}}\int_\Gamma \bar{p}^*_{\rho iq}u_i\,d\Gamma - \frac{\gamma E_{ve}}{E_e+E_{ve}}\int_\Gamma \bar{p}^*_{\rho iq}\dot{u}_i\,d\Gamma - \frac{\gamma E_e E_{ve}}{E_e+E_{ve}}C^{ke}_{\rho q}\dot{\varepsilon}_{ke}(p) + \frac{\gamma E_{ve}}{E_e+E_{ve}}\left[\int_\Gamma \bar{\sigma}^*_{\rho iq}\dot{p}_i\,d\Gamma + \dot{b}_i\int_\Gamma \overline{B}^*_{\rho iq}\,d\Gamma\right] + b_i\int_\Gamma \overline{B}^*_{\rho iq}\,d\Gamma + \frac{\gamma E_e E_{ve}}{E_e+E_{ve}}C^{ke}_{\rho q}\dot{\varepsilon}_{ke}(p) - \frac{\gamma E_{ve}}{E_e+E_{ve}}\dot{\sigma}_{\rho q}(p). \tag{B.11}$$

The forth and eighth terms at the right-hand side of Eq. (B.11) cancel each other, resulting:

$$\sigma_{\rho q}(p) = \int_\Gamma \bar{\sigma}^*_{\rho iq} p_i \,\mathrm{d}\Gamma - \frac{E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}} \int_\Gamma \bar{p}^*_{\rho iq} u_i \,\mathrm{d}\Gamma - \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}} \int_\Gamma \bar{p}^*_{\rho iq} \dot{u}_i \,\mathrm{d}\Gamma + \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}} \left[\int_\Gamma \bar{\sigma}^*_{\rho iq} \dot{p}_i \,\mathrm{d}\Gamma + \dot{b}_i \int_\Gamma \overline{B}^*_{\rho iq} \,\mathrm{d}\Gamma \right] + b_i \int_\Gamma \overline{B}^*_{\rho iq} \,\mathrm{d}\Gamma - \frac{\gamma E_{\mathrm{ve}}}{E_{\mathrm{e}} + E_{\mathrm{ve}}} \dot{\sigma}_{\rho q}(p). \tag{B.12}$$

Eq. (B.12) is the integral representation for total stress following the Boltzmann viscoelastic model. The functions $\bar{p}^*_{\rho iq}$, $\bar{\sigma}^*_{\rho iq}$ and $\overline{B}^*_{\rho iq}$ are given in Appendix C.

Appendix C. Some auxiliary values

The functions inside the kernels developed in this work are given by:

$$\hat{p}^*_{kie} = \frac{1}{4\pi(1-v)r^2} \left[\{2v(\delta_{ki} r_{,e} + \delta_{ei} r_{,k}) + 2\delta_{ek} r_{,i} - 8 r_{,k} r_{,i} r_{,e}\} \frac{\partial r}{\partial n} + (1-2v)(\delta_{ki} n_e + \delta_{ie} n_k - \delta_{ke} n_i + 2 r_{,k} r_{,e} n_i) + 2v(r_{,i} r_{,e} n_k + r_{,i} r_{,k} n_e) \right],$$

$$\varepsilon^*_{kie} = \frac{-1}{8\pi(1-v)Gr} \{(1-2v)(\delta_{ki} r_{,e} + \delta_{ke} r_{,i}) - \delta_{ie} r_{,k} + 2 r_{,k} r_{,i} r_{,e}\}, \tag{C.1}$$

$$\hat{B}^*_{kie} = \varepsilon^*_{kie} \frac{\partial r}{\partial n},$$

$$\bar{p}^*_{\rho iq} = \frac{2G}{4\pi(1-v)r^2} \left[2\frac{\partial r}{\partial n} \{(1-2v)\delta_{\rho q} r_{,i} + v(\delta_{\rho i} r_{,q} + \delta_{qi} r_{,\rho}) - 4 r_{,\rho} r_{,i} r_{,q}\} + 2v(n_\rho r_{,q} r_{,i} + n_q r_{,\rho} r_{,i}) + (1-2v)(2 n_i r_{,\rho} r_{,q} + n_q \delta_{\rho i} + n_\rho \delta_{qi}) - (1-4v) n_i \delta_{\rho q} \right], \tag{C.2}$$

$$\bar{\sigma}^*_{\rho iq} = \frac{1}{4\pi(1-v)r} \left[(1-2v)(\delta_{\rho q} r_{,i} + \delta_{iq} r_{,\rho} - \delta_{\rho i} r_{,q}) + 2 r_{,\rho} r_{,i} r_{,q} \right], \tag{C.3}$$

$$\overline{B}^*_{\rho iq} = \bar{\sigma}^*_{\rho iq} \frac{\partial r}{\partial n}. \tag{C.4}$$

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